Probabilistically reversible measurements

2001 J. Phys. A: Math. Gen. 34 9669

(http://iopscience.iop.org/0305-4470/34/45/307)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 9669–9676 PII: S0305-4470(01)26716-X

Probabilistically reversible measurements

Masashi Ban

Advanced Research Laboratory, Hitachi, Ltd, Higashi-Koigakubo 1-280, Kokubunji, Tokyo 185-8601, Japan

Received 5 July 2001, in final form 6 September 2001
Published 2 November 2001
Online at stacks.iop.org/JPhysA/34/9669

Abstract
Probabilistically reversible quantum measurements are considered from the information-theoretical point of view. It is found that an upper bound on the probability of a reversing quantum measurement is given by the generalized Bhattacharyya overlap of conditional probabilities of the measurement outcomes. The relation between the upper bound of the probability and the information obtained by the quantum measurement is also discussed.

PACS numbers: 03.65.Ta, 02.50.Cw, 03.67.−a

1. Introduction

Quantum measurement inevitably changes a quantum state of a measured physical system. A reversible quantum measurement is one in which by applying an appropriate quantum measurement, a measured physical system can recover its initial quantum state before the first quantum measurement was performed [1–6]. Reversible quantum measurement can be classified into three types. One is the completely reversible measurement in which the initial quantum state of the measured system can be deterministically recovered by applying an appropriate quantum measurement [1, 2]. Another is the probabilistically reversible measurement in which the physical system after the measurement can be restored to the initial quantum state with finite probability [3–5] and the last is the logically reversible measurement [6], where the initial quantum state can be calculated from the post-measurement state and the measurement outcome. If the logically reversible measurement is physically realizable, it is equivalent to the probabilistically reversible measurement. The characteristic of the completely reversible measurement is that we cannot obtain any information about the measured physical system. It has been shown that quantum teleportation [7] is the completely reversible measurement [1]. The information-theoretical properties of the completely reversible measurement have been investigated in detail [2]. The physical and operator-algebraic properties of the probabilistically reversible measurement have been studied [3–6], while the information-theoretical properties are not so clear. The information-theoretical approach to quantum measurement [8–10] is very important in relation to the recent development of the quantum information theory [11].

Therefore this paper investigates the information-theoretical properties of the probabilistically reversible measurement. It is shown that the upper bound on the probability of recovering the initial quantum state of the system before the first measurement is given by the generalized Bhattacharyya overlap of the conditional probabilities of the measurement outcomes, where the Bhattacharyya overlap of probabilities is the distinguishability measure of probabilities. Furthermore the relation between the upper bound on the probability of recovering the initial quantum state and the information obtained by the quantum measurement is also discussed. In section 2, the information-theoretical approach to quantum measurement is briefly reviewed. In section 3, the upper bound on the probability of the reversing measurement is obtained and the relation between the upper bound and the information gain is discussed. In section 4, to illustrate the result, the reversibility of the spin-$\frac{1}{2}$ measurement proposed by Royer [3, 4] is considered. In section 5, a summary is given.

2. Information-theoretical approach to quantum measurement

In this section, we briefly review a quantum measurement process from the information-theoretical point of view [8–10]. Suppose that we measure a value of some intrinsic observable $\hat{X}$, having a discrete spectrum, of a physical system, where we denote the projection-valued measure (or the spectral measure) of this observable, corresponding to the eigenvalue x, as $\hat{\mathcal{X}}(x) = |\psi(x)\rangle\langle\psi(x)|$ and the spectral set as Ω_X. Then the spectral decomposition of the observable $\hat{X}$ is given by $\hat{X} = \sum_{x\in\Omega_X} x\hat{\mathcal{X}}(x)$. Furthermore we denote the Hilbert space spanned by $\{|\psi(x)\rangle \mid x \in \Omega_X\}$ as $\mathcal{H}$.

To perform the quantum measurement, we first prepare a measurement apparatus and then we make an interaction between the physical system to be measured and the measurement apparatus to create some quantum correlation between them, where the unitary operator which describes the state change of the system–apparatus compound system is denoted as $\hat{U}$. The readout process of the result y exhibited by the measurement apparatus is described by a projection operator $\hat{\mathcal{Y}}^a(y) = |\phi^a(y)\rangle\langle\phi^a(y)|$, satisfying the relations $\hat{\mathcal{Y}}^a(y) \geqslant 0$ and $\sum_{y\in\Omega_Y} \hat{\mathcal{Y}}^a(y) = \hat{1}^a$, where Ω_Y is the set of all possible measurement outcomes and $\hat{1}^a$ is an identity operator defined on the Hilbert space of the measurement apparatus. If the quantum state of the relevant system before the interaction with the measurement apparatus is given by $\hat{\rho}_{\text{in}}$ and the measurement apparatus is prepared in a quantum state $|0^a\rangle$, the quantum state of the compound system just before the readout of the measurement outcome is given by $\hat{W}_{\text{out}} = \hat{U}(\hat{\rho}_{\text{in}} \otimes |0^a\rangle\langle 0^a|)\hat{U}^\dagger$. Then the probability $P^A_{\text{out}}(y)$ of obtaining the measurement outcome y is given by

$$P_{\text{out}}(y) = \text{Tr}_S[\hat{A}^\dagger(y)\hat{A}(y)\hat{\rho}_{\text{in}}] \tag{1}$$

where the operator $\hat{A}(y)$ of the system depends on the system–apparatus interaction, the initial state of the apparatus and the readout process of the measurement

$$\hat{A}(y) = \langle\phi^a(y)|\hat{U}|0^a\rangle \tag{2}$$

which satisfies the normalization condition

$$\sum_{y\in\Omega_Y} \hat{A}^\dagger(y)\hat{A}(y) = \hat{1}. \tag{3}$$

The identity operator $\hat{1}$ is defined on the Hilbert space $\mathcal{H}$. The quantum state $\hat{\rho}_{\text{out}}(y)$ of the physical system after obtaining the measurement outcome y is given by the state-reduction formulae [12–14]

$$\hat{\rho}_{\text{out}}(y) = \frac{\hat{A}(y)\hat{\rho}_{\text{in}}\hat{A}^\dagger(y)}{\text{Tr}\,[\hat{A}^\dagger(y)\hat{A}(y)\hat{\rho}_{\text{in}}]}. \tag{4}$$

The set of operators $\{\hat{A}(y) \mid y \in \Omega_Y\}$ completely characterizes the quantum measurement process. Since we can always exclude the values y that can never be obtained by the quantum measurement, in the rest of this paper we assume that $P_{\rm out}(y) \neq 0$ for all $y \in \Omega_Y$.

In the previous papers [8–10], we have investigated the information-theoretical properties of quantum measurement processes and we have obtained the necessary and sufficient condition that the amount of information about the intrinsic observable $\hat{X}$, obtained from the measurement outcome, can be represented by the Shannon mutual information. The condition is that the intrinsic observable $\hat{\mathcal{X}}(x)$ (or $\hat{X}$) of the physical system commutes with the operational observable given by $\hat{A}^\dagger(y)\hat{A}(y)$, i.e.

$$[\hat{\mathcal{X}}^S(x), \hat{A}^\dagger(y)\hat{A}(y)] = 0 \qquad (\forall\, x \in \Omega_X \quad \forall\, y \in \Omega_Y). \tag{5}$$

It is found that this condition is satisfied by many kinds of quantum measurements such as the photon counting process, the standard position (momentum) measurement and the quantum nondemolition measurement [8, 9]. If the condition (5) is satisfied, the amount of information about the intrinsic observable of the physical system obtained by the quantum measurement is given by

$$I(Y:X) = \sum_{x\in\Omega_X}\sum_{y\in\Omega_Y} P(y|x)P_{\rm in}(x)\log\left[\frac{P(y|x)}{P_{\rm out}(y)}\right] \tag{6}$$

where $P_{\rm in}(x) = \langle\psi(x)|\hat{\rho}_{\rm in}|\psi(x)\rangle$ is the probability that the intrinsic observable $\hat{X}$ takes a value x in the initial quantum state $\hat{\rho}_{\rm in}$ of the system and the conditional probability $P(y|x)$ that the measurement outcome y is obtained when the intrinsic observable takes a value x is given by

$$P(y|x) = \langle\psi(x)|\hat{A}^\dagger(y)\hat{A}(y)|\psi(x)\rangle \tag{7}$$

which connects the initial probability $P_{\rm in}(x)$ with the output probability $P_{\rm out}(y)$

$$P_{\rm out}(y) = \sum_{x\in\Omega_X} P(y|x)P_{\rm in}(x). \tag{8}$$

Furthermore, from equations (5) and (7) we obtain the relation between the intrinsic and operational observables:

$$\hat{A}^\dagger(y)\hat{A}(y) = \sum_{x\in\Omega_X} P(y|x)\hat{\mathcal{X}}(x). \tag{9}$$

The Shannon mutual information $I(Y:X)$ given by equation (6) represents how much classical information about the intrinsic observable $\hat{\mathcal{X}}^S(x)$ is transmitted from the physical system before performing the quantum measurement to the observer who performed the quantum measurement. The mutual information satisfies the inequality $0 \leqslant I(Y:X) \leqslant H(X)$, where $H(X)$ is the Shannon entropy of the intrinsic observable in the initial quantum state $\hat{\rho}_{\rm in}$ of the physical system, that is, $H(X) = -\sum_{x\in\Omega_X} P_{\rm in}(x)\log P_{\rm in}(x)$.

Suppose that the intrinsic observable $\hat{\mathcal{X}}(x)$ of the physical system commutes with the unitary operator $\hat{U}$ which describes the state change caused by the system–apparatus interaction. For example, such a condition is satisfied by any quantum nondemolition measurement. In this case, we obtain $[\hat{\mathcal{X}}(x), \hat{A}(y)] = 0$ from equation (2). Combined with equation (9), this commutativity yields

$$\hat{A}(y) = \sum_{x\in\Omega_X} \sqrt{P(y|x)}\hat{\mathcal{X}}(x) \tag{10}$$

where $\hat{\mathcal{X}}(x)\hat{\mathcal{X}}(x') = \delta_{xx'}\hat{\mathcal{X}}(x)$ has been used and an unimportant phase factor has been ignored. On the other hand, when the Hilbert space $\mathcal{H}$ is a finite-dimensional space, the polar decomposition theorem for operators says that there is a unitary operator $\hat{V}(y)$ such that $\hat{A}(y) = \hat{V}(y)\sqrt{\hat{A}^\dagger(y)\hat{A}(y)}$. Thus from equation (9) we have the relation

$$\hat{A}(y) = \hat{V}(y) \sum_{x\in\Omega_X} \sqrt{P(y|x)}\hat{\mathcal{X}}(x). \tag{11}$$

In this case, the measurement process can be divided into two processes [15]. One is called the raw measurement, which is described by the set of non-negative operators $\left\{\sqrt{\hat{A}^\dagger(y)\hat{A}(y)} \mid y \in \Omega_Y\right\}$. The other is called the feedback process, which is described by the unitary operator $\hat{V}(y)$ depending on the measurement outcome y. The quantum measurement without the feedback process plays an important role in the quantum information processing [15].

When the quantum measurement is described by the operator $\hat{A}(y)$ without the feedback process, the post-measurement state $\hat{\rho}_{\text{out}}(y)$ of the physical system after obtaining the measurement outcome y becomes

$$\hat{\rho}_{\text{out}}(y) = \frac{\sum_{x\in\Omega_X}\sum_{x'\in\Omega_X}\sqrt{P(y|x)P(y|x')}\hat{\mathcal{X}}(x)\hat{\rho}_{\text{in}}\hat{\mathcal{X}}(x')}{\sum_{x\in\Omega_X} P(y|x)P_{\text{in}}(x)} \tag{12}$$

with the probability $P_{\text{out}}(y) = \sum_{x\in\Omega} P(y|x)P_{\text{in}}(x)$. In this state, the observable $\hat{X}$ takes the value x with the probability $P(x|y) = \text{Tr}\,[\hat{\mathcal{X}}(x)\hat{\rho}_{\text{out}}(y)]$. Then the average value of the Shannon entropy of the post-measurement state $\hat{\rho}_{\text{out}}(y)$ is calculated to be

$$\begin{aligned} H(X|Y) &= -\sum_{y\in\Omega_Y}\sum_{x\in\Omega_X} P_{\text{out}}(y)P(x|y)\log P(x|y) \\ &= -\sum_{y\in\Omega_Y}\sum_{x\in\Omega_X} P(y|x)P_{\text{in}}(x)\log\left[\frac{P(y|x)P_{\text{in}}(x)}{P_{\text{out}}(y)}\right] \\ &= I(Y:X) + H(X) \end{aligned} \tag{13}$$

where $I(Y:X)$ is the amount of information about the observable $\hat{X}$ obtained by the quantum measurement and $H(X)$ is the Shannon entropy of the initial quantum state of the system. Therefore it is easy to see that the average decrease of the Shannon entropy of the system, $\Delta H(Y:X) \equiv H(X) - H(X|Y)$, is equal to the information gain $I(Y:X)$ by the quantum measurement, that is,

$$\Delta H(Y:X) = I(Y:X). \tag{14}$$

This result indicates that the quantum measurement described by the operator $\hat{A}(y)$ given by equation (10) conserves the information on the observable $\hat{X}$. Furthermore, when no information is obtained by the quantum measurement, namely, $I(Y:X) = 0$, the conditional probability $P(y|x)$ is independent of x. Then it is easy to see from equation (12) that the post-measurement state $\hat{\rho}_{\text{out}}(y)$ becomes equal to the initial state $\hat{\rho}_{\text{in}}$ of the system, where the relation $\sum_{x\in\Omega_X}\hat{\mathcal{X}}(x) = \hat{1}$ is used. This result is closely related to quantum teleportation [1].

3. Probability of the reversing measurement

In this section, we consider the reversibility of quantum measurements that a measured physical system can be restored to the initial state with finite probability. Such a reversing process is performed by an appropriate quantum measurement followed by a unitary transformation. Suppose that an operator $\hat{R}_y$ describes the reversing measurement after the measurement outcome y was obtained in the first measurement. The condition that the quantum measurement described by the operator $\hat{R}_y$ restores the post-measurement state $\hat{\rho}_{\text{out}}(y)$ to the initial state $\hat{\rho}_{\text{in}}$ of the system is given by

$$\hat{R}_y\hat{A}(y) = m_y\hat{U}_y \tag{15}$$

where $\hat{U}_y$ is a unitary operator and m_y is some non-zero c number, both of which depend on the measurement outcome y. Then we have

$$\hat{U}_y^\dagger\left(\frac{\hat{R}_y\hat{\rho}_{\text{out}}(y)\hat{R}_y^\dagger}{\text{Tr}\,[\hat{R}_y^\dagger\hat{R}_y\hat{\rho}_{\text{out}}(y)]}\right)\hat{U}_y^\dagger = \hat{\rho}_{\text{in}}. \tag{16}$$

Here it is important to note that the unitarity of the operator $\hat{U}_y$ requires $\hat{A}(y)|\psi\rangle \neq 0$ $(\forall|\psi\rangle \in \mathcal{H})$. Since the state change $\hat{\rho}_{\text{out}}(y) \to \hat{\rho}_{\text{in}}$ occurs with probability $\text{Tr}\,[\hat{R}_y^\dagger\hat{R}_y\hat{\rho}_{\text{out}}(y)]$, the average probability $\bar{P}$ of the reversing quantum measurement is given by

$$\begin{aligned}\bar{P} &= \sum_{y\in\Omega_Y} P_{\text{out}}(y)\,\text{Tr}\,[\hat{R}_y^\dagger\hat{R}_y\hat{\rho}_{\text{out}}(y)] \\ &= \sum_{y\in\Omega_Y} m_y^2\end{aligned} \tag{17}$$

where $P_{\text{out}}(y)$ is given by equation (1). The condition that the reversing process can be physically realized is $\hat{R}_y^\dagger\hat{R}_y \leqslant \hat{1}$.

We now obtain the upper bound on the probability $\bar{P}$ of the reversing measurement. The physical realizability condition of the reversing measurement, $\hat{R}_y^\dagger\hat{R}_y \leqslant \hat{1}$, yields

$$\begin{aligned}\langle\phi|\hat{A}^\dagger(y)\hat{A}(y)|\phi\rangle &\geqslant \langle\phi|\hat{A}^\dagger(y)\hat{R}_y^\dagger\hat{R}_y\hat{A}(y)|\phi\rangle \\ &= m_y^2\langle\phi|\phi\rangle\end{aligned} \tag{18}$$

for any vector $|\phi\rangle \in \mathcal{H}$, where equation (15) has been used. Hence the parameter m_y satisfies the inequality

$$m_y^2 \leqslant \inf_{|\phi\rangle\in\mathcal{H}} \frac{\langle\phi|\hat{A}^\dagger(y)\hat{A}(y)|\phi\rangle}{\langle\phi|\phi\rangle}. \tag{19}$$

Then we obtain from equation (9)

$$\begin{aligned}m_y^2 &\leqslant \inf_{|\phi\rangle\in\mathcal{H}} \sum_{x\in\Omega_Y} P(y|x)\frac{\langle\phi|\hat{\mathcal{X}}(x)|\phi\rangle}{\langle\phi|\phi\rangle} \\ &= \inf_{|\phi\rangle\in\mathcal{H}} \sum_{x\in\Omega_Y} P(y|x)\frac{|\langle\psi(x)|\phi\rangle|^2}{\langle\phi|\phi\rangle} \\ &= \min_{x\in\Omega_X} P(y|x).\end{aligned} \tag{20}$$

Thus the average probability $\bar{P}$ of the reversing quantum measurement satisfies the inequality

$$\bar{P} \leqslant \sum_{y\in\Omega_Y}\big[\min_{x\in\Omega_X} P(y|x)\big]. \tag{21}$$

If the spectral set Ω_X of the intrinsic observable is a finite set, we obtain

$$\bar{P} \leqslant \sum_{\Omega_Y}\left[\prod_{x\in\Omega_X} P(y|x)\right]^{1/|\Omega_X|} \equiv B_{|\Omega_X|}(Y:X) \tag{22}$$

where we have used the inequality $\min_{1\leqslant j\leqslant n} a_j \leqslant (a_1a_2\cdots a_n)^{1/n}$ ($a_j \geqslant 0$ for all j) and $|\Omega_X|$ is the number of the elements of the spectral set Ω_X.

The parameter $B_{|\Omega_X|}(Y:X)$ satisfies the following properties. If the equality $P(y|x) = P(y|x')$ holds for all $x, x' \in \Omega_X$, we have $B_{|\Omega_X|}(Y:X) = 1$. On the other hand, the parameter $B_{|\Omega_X|}(Y:X)$ vanishes if and only if for each $y \in \Omega_Y$ there exists $x \in \Omega_X$, depending on y, such that $P(y|x) = 0$. Furthermore, using the fact that the geometric mean is not greater than the arithmetic mean, namely $(a_1 a_2 \cdots a_n)^{1/n} \leqslant (1/n)(a_1 + a_2 + \cdots + a_n)$ $(a_j \geqslant 0)$, and using the normalization condition of the conditional probability $\sum_{y\in\Omega_Y} P(y|x) = 1$, we obtain the inequality

$$0 \leqslant B_{|\Omega_X|}(Y:X) \leqslant 1. \tag{23}$$

When the observable $\hat{X}$ is defined on a two-dimensional Hilbert space, we have $|\Omega_X| = 2$ and we can set $\Omega_X = \{0, 1\}$. In this case, the upper bound on the probability of the reversing quantum measurement becomes

$$B_2(Y:X) = \sum_{y\in\Omega_Y} \sqrt{P(y|0)P(y|1)}. \tag{24}$$

This is the Bhattacharyya overlap of the conditional probabilities $P(y|0)$ and $P(y|1)$, which is the distinguishability measure of these probabilities [16]. As the conditional probabilities $P(y|x)$ and $P(y|x')$ $(x \neq x')$ is more distinguishable, the upper bound $B_{|\Omega_X|}(Y:X)$ is smaller and the information gain $I(Y:X)$ is greater. In the most distinguishable case, where $P(y|x) = \delta_{xy}$, we have $B_{|\Omega_X|}(Y:X) = 0$ and $I(Y:X) = \log|\Omega_X|$, while in the least distinguishable case, where $P(y|x)$ is independent of x, we have $B_{|\Omega_X|}(Y:X) = 1$ and $I(Y:X) = 0$. The former means that when we obtain the maximum information about the intrinsic observable, the initial quantum state of the system cannot be recovered, and the latter corresponds to the completely reversible measurement [1,2]. In the following, we refer to the parameter $B_{|\Omega_X|}(Y:X)$ as the generalized Bhattacharyya overlap. Therefore the upper bound on the average probability of the reversing quantum measurement is given by the generalized Bhattacharyya overlap of the conditional probabilities of the measurement outcomes.

4. Reversibility of the spin-$\frac{1}{2}$ measurement

To illustrate the result obtained in the previous section, we consider the probabilistically reversible spin-$\frac{1}{2}$ measurement investigated by Royer [3]. In this measurement, the unitary operator $\hat{U}$ which describes the state change due to the system–apparatus interaction is given by $\hat{U} = \exp(-\mathrm{i}\sigma\hat{S}_z \otimes \hat{S}^a_y)$ [17], where the operator $\hat{S}_z$ is a z component of spin-$\frac{1}{2}$ of the system, the operator $\hat{S}^a_y$ is a y component of spin-$\frac{1}{2}$ of the apparatus, and σ is some real parameter. The measurement apparatus before the interaction with the system is prepared in a superposition state $|0^a\rangle = \cos(\frac{1}{2}\theta)|\uparrow^a\rangle + \sin(\frac{1}{2}\theta)|\downarrow^a\rangle$, where $\hat{S}^a_z|\uparrow^a\rangle = \frac{1}{2}|\uparrow^a\rangle$ and $\hat{S}^a_z|\downarrow^a\rangle = -\frac{1}{2}|\downarrow^a\rangle$. The readout of the measurement result is the Stern–Gerlach measurement of the z component of the spin-$\frac{1}{2}$ particle of the measurement apparatus. Then the measurement operators $\hat{A}(0)$ and $\hat{A}(1)$ are given by

$$\hat{A}(0) = \cos(\tfrac{1}{2}\theta + \tfrac{1}{4}\sigma)\hat{\mathcal{X}}(0) + \cos(\tfrac{1}{2}\theta - \tfrac{1}{4}\sigma)\hat{\mathcal{X}}(1) \tag{25}$$

$$\hat{A}(1) = \sin(\tfrac{1}{2}\theta + \tfrac{1}{4}\sigma)\hat{\mathcal{X}}(0) + \sin(\tfrac{1}{2}\theta - \tfrac{1}{4}\sigma)\hat{\mathcal{X}}(1) \tag{26}$$

where $\hat{\mathcal{X}}(0) = |\uparrow\rangle\langle\uparrow|$ and $\hat{\mathcal{X}}(1) = |\downarrow\rangle\langle\downarrow|$ with $\hat{S}_z|\uparrow\rangle = \frac{1}{2}|\uparrow\rangle$ and $\hat{S}_z|\downarrow\rangle = -\frac{1}{2}|\downarrow\rangle$. The conditional probability $P(j|k)$ of the measurement, given by (7), is expressed as a 2×2 matrix

$$\boldsymbol{P} = \begin{pmatrix} \cos^2(\frac{1}{2}\theta + \frac{1}{4}\sigma) & \cos^2(\frac{1}{2}\theta - \frac{1}{4}\sigma) \\ \sin^2(\frac{1}{2}\theta + \frac{1}{4}\sigma) & \sin^2(\frac{1}{2}\theta - \frac{1}{4}\sigma) \end{pmatrix}. \tag{27}$$

The upper bound $B_2(Y:X)$ on the average probability $\bar{P}$ of the reversing measurement is easily obtained:

$$B_2(Y:X) = |\cos(\tfrac{1}{2}\theta + \tfrac{1}{4}\sigma)\cos(\tfrac{1}{2}\theta - \tfrac{1}{4}\sigma)| + |\sin(\tfrac{1}{2}\theta + \tfrac{1}{4}\sigma)\sin(\tfrac{1}{2}\theta - \tfrac{1}{4}\sigma)|. \tag{28}$$

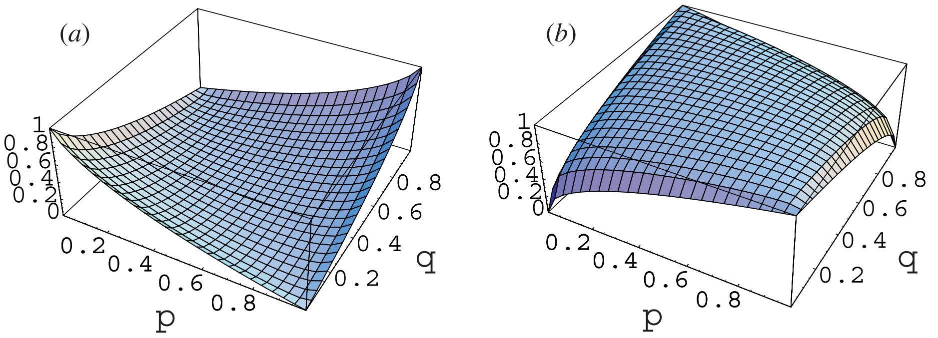

Figure 1. The plots of (*a*) the information gain $I(p, q)$ and (*b*) the upper bound $B(q, p)$, where the information is measured in *bits* and $a = \frac{1}{2}$ is assumed.

(This figure is in colour only in the electronic version)

In particular, if the parameters θ and σ are chosen such that $\cos(\frac{1}{2}\theta \pm \frac{1}{4}\sigma) \geqslant 0$ ($\leqslant 0$) and $\sin(\frac{1}{2}\theta \pm \frac{1}{4}\sigma) \geqslant 0$ ($\leqslant 0$), the upper bound is simplified as $B_2(Y:X) = \cos^2(\frac{1}{2}\sigma)$.

To investigate the relation between the upper bound $B_2(Y:X)$ on the average probability of the reversing quantum measurement and the information gain $I(Y:X)$ by performing the quantum measurement, we set $P(0|0) = 1-P(1|0) = 1-p$ and $P(1|1) = 1-P(0|1) = 1-q$ with $0 \leqslant p \leqslant 1$ and $0 \leqslant q \leqslant 1$. In this case, the upper bound $B(p,q)$ becomes

$$B(p,q) = \sqrt{(1-p)q} + \sqrt{p(1-q)} \tag{29}$$

and the information $I(p,q)$ obtained by the quantum measurement is given by

$$\begin{aligned} I(p,q) = {} & (1-p)(1-a)\log\left[\frac{1-p}{(1-p)(1-a)+qa}\right] \\ & + qa\log\left[\frac{q}{(1-p)(1-a)+qa}\right] \\ & + p(1-a)\log\left[\frac{p}{p(1-a)+(1-q)a}\right] \\ & + (1-q)a\log\left[\frac{1-q}{p(1-a)+(1-q)a}\right] \end{aligned} \tag{30}$$

where we set $\langle\uparrow|\hat{\rho}_{\rm in}|\uparrow\rangle = 1 - \langle\downarrow|\hat{\rho}_{\rm in}|\downarrow\rangle = 1-a$ with $0 \leqslant a \leqslant 1$. The upper bound $B(p,q)$ and the information gain $I(p,q)$ are plotted in figure 1.

This figure clearly shows that, as the information gain in the quantum measurement is greater, the possibility that we can recover the initial quantum state of the system becomes smaller.

5. Conclusion

In this paper, we have investigated the information-theoretical properties of the probabilistically reversible measurement and we have found that the upper bound on the average probability of the reversing measurement is given by the generalized Bhattacharyya overlap $B_{|\Omega_X|}(Y:X)$ of the conditional probabilities of the measurement outcomes. Furthermore we have shown the relation between the information obtained by the quantum measurement and the probability of recovering the initial quantum state of the system.

References

[1] Nielsen M A and Caves C M 1997 *Phys. Rev.* A **55** 2547
[2] Nielsen M A, Caves C M, Schumacher B and Bernum H 1998 *Proc. R. Soc.* A **454** 277
[3] Royer A 1994 *Phys. Rev. Lett.* **73** 913
[4] Royer A 1995 *Phys. Rev. Lett.* **74** 1040 (erratum)
[5] Koashi M and Ueda M 1999 *Phys. Rev. Lett.* **82** 2598
[6] Ueda M, Imoto N and Hasegawa H 1996 *Phys. Rev.* A **53** 3808
[7] Bennett C H, Brassard G, Crepeau C, Jozsa R, Peres A and Wootters W K 1993 *Phys. Rev. Lett.* **70** 1895
[8] Ban M 1998 *Int. J. Theor. Phys.* **37** 2491
[9] Ban M 1999 *J. Phys. A: Math. Gen.* **32** 1643
[10] Ban M 1999 *J. Phys. A: Math. Gen.* **32** 6527
[11] Nielsen M A and Chuang I L 2000 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press)
[12] Davies E B 1976 *Quantum Theory of Open Systems* (New York: Academic)
[13] Kraus K 1983 *States, Effects, and Operations* (Berlin: Springer)
[14] Ozawa M 1984 *J. Math. Phys.* **25** 79
[15] Fuchs C A and Jacobs K 2001 *Phys. Rev.* A **63** 062305
[16] Cover T M and Thomas J A 1991 *Elements of Information Theory* (New York: Wiley)
[17] Braunstein S L and Caves C M 1988 *Found. Phys. Lett.* **1** 1